

Pricing of Futures Contracts by Considering Stochastic Exponential Jump Domain of Spot Price

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Abstract

Derivatives are alternative financial instruments which extend traders opportunities to achieve some financial goals. They are risk management instruments that are related to a data in the future, and also they react to uncertain prices. Study on pricing futures can provide useful tools to understand the stochastic behavior of prices to manage the risk of price volatility. Thus, this study evaluates commodity futures contracts by considering Ross (1995) one-factor future pricing model as a function of spot price, Gibson and Schwartz (1990) two-factor futures pricing model as a function of spot price and convenience yield and finally Schwartz (1997) three-factor futures pricing model as a function of spot price, convenience yield and instantaneous interest rate by adding jump to stochastic behavior of commodity spot price. For this purpose, it is assumed that spot price follows Jump-diffusion stochastic process with exponential probability distribution of jump domain. Finally, commodity pricing future relations in three basic models are presented as a function of above factor(s) and jump parameters by using Duffy-Pan-Singleton approach.

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1. Introduction

Derivatives are a tool for firms and other users to reduce risk of price volatility. A derivative is a financial instrument that has a value determined by the price of underlying asset (McDonald, 2006). Overall, derivatives can be divided into four groups: Forward Contract, Futures Contract, Option Contract, and Swap Contract.

The Use of these tools to manage a risk depends on the determination of delivery price at the time of signing contract. Correct pricing in derivative market for this reason is important because they are used to control for the underlying asset price volatility risk. Hence, traders expect the determined delivery prices can reduce the spot prices volatility. Now, the question is that if trader sells or buys derivative instruments with a specific maturity date, which price he (she) should receive or pay?

Theoretical research of derivatives pricing was developed by efforts of Merton (1973) and Black and Scholes (1973). The approach, which is used by them, presents the price relations for European option contract. This relationship and how to extract them plays a pivotal role in the pricing of other derivatives, particularly option and futures contracts that would be born in next decades.

In this study, the theoretical pricing of commodity futures contract is considered as three models of one-factor, two-factor and three-factor pricing that are presented by using Exponential probability distribution of jump domain in underlying commodity spot price. Due to the relationship between futures price and spot price, it is expected that the spot price jumps have significant effect on futures price.

With Exponential jump domain, one-factor model is presented by using Ross (1995) one-factor futures pricing model as a function of spot price. The Gibson-Schwartz (1990) and Schwartz (1997) futures pricing model as a function of spot price and convenience yield are also used to present two-factor model. In addition, the three-factor model is analyzed through the Schwartz (1997) three-factor futures pricing model as a function of spot price, convenience yield and instantaneous interest rate. In this paper, all three models are reviewed by assuming adherence spot price to jump-diffusion stochastic process. According to this, it is assumed that spot price jump domain follows exponential probability distribution.

Considering above assumption in basic models, a futures price equations are extracted by Duffie-Pan-Singleton (2000) approach. In specified price equations, commodity futures price would be a function of above factor(s) and

jump parameters. The rest of the paper stands for literature reviews in Section 2, a theoretical framework in Section 3 and conclusion in Section 4.

2. Literature Review

Studies conducted on the pricing of derivatives are classified into several types of derivative instruments. Schwartz and Smith (2000) in their study provide a two-factor model to commodity futures pricing. In this article, spot price in short term follows mean reversion process and the long-term equilibrium prices in the model are stochastic. Both of two presented factors in model are invisible but would be estimated by futures prices. In this model, price volatility in contracts with longer maturities provides required information of equilibrium price and information of short-term price volatility is covered by the difference between the long-term equilibrium price and the spot price. This model does not use obviously convenience yield, while Schwartz and Smith believe that their long-term and short-term model is equivalent to stochastic convenience yield which is presented by Gibson-Schwartz (1990). Parameters of the model are estimated by using Kalman filter approach on oil futures price data.

Villaplana (2003) studied behavior of prices in electricity futures contract. He provided a two-factor model by using seasonal data of commodity futures contract. The model is further development of Schwartz and Smith (2000) and Lucia-Schwartz (2002). It shows the main contribution of the paper is the inclusion of a jump component, with a non-constant probability of occurrence of jumps, in a short-term factor. In this study, the stochastic behavior of unobservable variables is modeled by Affine diffusion and Affine jump-diffusion and finally futures contract prices is presented.

Carmona and Ludkovski (2004) in "Spot convenience yield for the energy markets" reviewed the models that using convenience yield as an effective factor on energy futures prices. From a mathematical point of view, convenience yield is obtained by studying dynamic models in the valuation of commodity prices. Since convenience yield is directly invisible factor, stochastic filtering is used as a approach to estimate it via visible variables. Due to this subject, the results of the study indicate inconsistency of futures pricing models in using convenience yield.

Aiube and Samanez (2014) compare Schwartz-Smith (2000) and Schwartz(1997) two and three factor models on commodity futures pricing to show advantage and disadvantage of

both models. As previously discussed, futures price is determined by short-run price variations and long-run equilibrium price in Schwartz-Smith (2000) model. Schwartz Three-factor model considers dynamics of the spot price, interest rate and convenience yield to futures pricing. Schwartz and Smith (2000) indicate that the three- factor Schwartz model in comparison with the two- factor model has a better performance in long-run pricing. Aiube and Samanez (2014) in their paper again compare these two models by long-term and short-term data. The results of this comparison indicate that three- factor model has better performance in determination of time structure of futures price with longer maturities.

Dempster et al. (2015) investigate long and short term jumps in commodity futures prices and consider this issue from economic and stochastic point of view. They study daily changes in oil and copper futures prices to show that both of commodity spot price and convenience yield in these markets are so volatile and have a number of stochastic jumps. Therefore, they add separately short-term and long-term jumps on stochastic behavior of spot price in Schwartz and Smith (2000) model to extract commodity futures price relations. Finally, with regard to an empirical model for oil and copper futures markets, they use Importance State Space to estimate of parameters. Estimated parameters in the empirical studies, show that futures pricing models with jumps have better performance than models without jump

3. Theoretical Framework

3.1. One-factor model of futures pricing with spot price jump (exponential jump domain)

Increased interest to use of this model in 1970s revealed the shortcomings of Black-Scholes model. According to this approach, many efforts made to modify Black-Scholes model. Martin (2007) finds futures price by easing the assumption of continuity of spot price behavior in Black-Scholes model. In this context, models with jump in stochastic variables such as spot prices are considered. Some of these models assume price changes are continuous being are associated with jump in a few cases. These models are known as jump-diffusion processes. The term of diffusion refers to the continuous trend of price that can be explained through the standard Brownian process. The process of jump-diffusion is a set of a drift term, a Brownian motion and a compound Poisson process (Tankov and Rama, 2004: 265).

In this section, Ross and Schwartz one-factor futures pricing model is developed by adding

jump into commodity spot price behavior. For this purpose, we assume that (Ω, \mathcal{F}, P) is a probability space and the model is presented in this space. Ω is sample space, \mathcal{F} is a sigma field of Ω subsets that represents the accumulated information flow and P is a probability measure function with maximum amount of 1 on \mathcal{F} .

In order to extract the futures price $F(T, S_t)$ with futures time maturity T , it is assumed that the commodity spot price (S_t) has increasing and decreasing jump terms that are added separately to diffusion component. Therefore, the spot price of the commodity follows jump-diffusion process as:

$$dS_t = \kappa(\beta - \ln S_t)S_t dt + \sigma dB_t + S_t(e^{J_u} - 1)dN_{u,t} + S_t(e^{-J_d} - 1)dN_{d,t} \quad (1)$$

In equation (1), $\kappa(\beta - \ln S_t)S_t dt$ presents the time trend of commodity spot price (S_t) that κ and β are parameters. σdB_t is spot price volatility that σ is parameter and B_t is Standard Brownian motion. $S_t(e^{J_u} - 1)dN_{u,t}$ represents the process of upward or increasing net jump that follows the Poisson process $N_{u,t}$ with rate η_u . Also the domain of increasing stochastic spot price jumps is shown by J_u that follows the exponential probability distribution with rate parameter γ_u . $S_t(e^{-J_d} - 1)dN_{d,t}$ is a downward or decreasing net jump process with a Poisson process $N_{d,t}$ with rate η_d . Also, the absolute value of the domain of spot price stochastic reduction jump is shown by J_d that follows the exponential probability distribution with rate parameter γ_d . In this model, B_t , $dN_{u,t}$, $dN_{d,t}$, J_u and J_d are independent.

Assuming that X_t is natural logarithm of spot price S_t , i.e. $X_t = \ln S_t$. By using jump-diffusion Ito's lemma for (1) it can be written as:

$$dX_t = \kappa(\mu - X_t)dt + \sigma dB_t + J_u dN_{u,t} - J_d dN_{d,t} \quad (2)$$

$$\mu = \beta - \frac{\sigma^2}{2\kappa}$$

Based on this relationship, logarithm of spot price will follow Mean Reversion jump-diffusion Process where μ represents the mean of log spot price, $\kappa > 0$ is speed of adjustment, σ is the standard deviation from the mean value and B_t is Standard Brownian Process. Mean reversion process or Ornstein-Uhlenbeck Process is first time utilized by Vasicek (1977) to model the behavior of interest rate. The second term of Equation (2) shows the spot price volatility where dB_t represents increment of standard Brownian process. The third and four term of

equation show the increasing and decreasing jump component of commodity spot price.

In order to extract the relationship of risk neutral price $F(T, S_t)$, using of Equivalent Martingale Measure Q is required and in order to converse the target probability measure P into the equivalent probability measure Q is used of Girsanov conversion $dB_t = Mdt + dB_t^*$ in Equation (2).

$$dX_t = \kappa(\mu^* - X_t)dt + \sigma dB_t^* + J_u dN_{u,t} - J_d dN_{d,t} \quad (3)$$

$$\mu^* = \mu + \frac{\sigma M}{\kappa} = \mu - \lambda$$

where λ is the market value of risk for each unit of X_t and it is assumed constant. dB_t^* also represents the increment of Standard Brownian Process with regard to the equivalent probability measure. Also, μ^* is the mean value in probability space Q .

Using the previous equation that shows the dynamics of log spot price in equivalent probability space, the closed form answer of futures price $F(T, S_t)$ is obtained by using of Daffie-Pan-Singleton approach and application of Affine functions. The function $f(x_1, \dots, x_2)$ is called Affine function if it can be written in a linear form as follows:

$$f(x_1, \dots, x_2) = A_1 x_1 + \dots + A_n x_n + b \quad (4)$$

Affine function is a combination of a linear function with coefficients A_i and a fixed amount b . In order to use this function, jump-diffusion process Z_t is written in n -dimension case as follows:

$$dZ_t = \mu(Z_t)dt + \sigma(Z_t)dB_t + h(J)dN_t \quad (5)$$

where:

$$\begin{aligned} \mu(t, Z_t): \mathbb{R}_+ \times \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ C(t, Z_t) = \sigma(t, Z) \sigma'(t, Z): \mathbb{R}_+ \times \mathbb{R}^n &\rightarrow M(n, n) \end{aligned}$$

where $M(n, n)$ is a $n \times n$ matrix, $\sigma'(t, Z)$ is the transposed of $\sigma(t, Z): \mathbb{R}_+ \times \mathbb{R}^n \rightarrow M(n, d)$ and dB_t is the standard Brownian process in \mathbb{R}^d .

The above mentioned jump-diffusion process is called Affine jump-diffusion process if $\mu(t, Z_t)$, $C(t, Z_t)$ and the jump rate $\eta(Z_t)$ are Affine functions of Z_t :

$$\begin{aligned} \mu_i(t, Z_t) &= d_{i0} + d_{i1}Z_1 + \dots + d_{in}Z_n \\ C(t, Z_t) &= e_0 + e_1Z_1 + \dots + e_nZ_n \\ \eta(Z_t) &= m_0 + m_1Z_1 + \dots + m_nZ_n \quad (6) \\ d_{ij} \in \mathbb{R}, \quad i &= 1, \dots, n, \quad j = 0, \dots, n, \\ e_j \in \mathbb{R}^{n \times n} \quad \text{and} \quad m_j &\in \mathbb{R}^+ \end{aligned}$$

According to the structure of Equation (3), this equation follows the Affine jump-diffusion

process. Characteristic function of stochastic vector X_t is obtained by using of these functions. Affine jump-diffusion process of Equation (3) has the Exponentially Affine characteristic function related to maturity T and X_t as follows:

$$\mathbb{E}^Q [e^{uX_T} | \mathcal{F}_t] = \exp(\phi(T-t, u) + X_t \psi(T-t, u)) \quad (7)$$

where the characteristic function is expressed as conditional expect e^{uX_T} with respect to the flow of information \mathcal{F}_t in equivalent probability space Q where $iz = u \in i\mathbb{R}$. To achieve $\phi(T-t, u)$ and $\psi(T-t, u)$ is used follows Riccati equation:

$$\begin{aligned} \dot{\phi}(T-t, u) &= \kappa \mu^* \psi + \frac{1}{2} \sigma^2 \psi^2 + \eta_u k_u(\psi) \\ &\quad + \eta_d k_d(-\psi), \quad (8) \\ \phi(0, u) &= 0 \end{aligned}$$

$$\begin{aligned} \dot{\psi}(T-t, u) &= -\kappa \psi \Rightarrow \psi(T-t, u) \\ &= u e^{-\kappa(T-t)} \quad (9) \end{aligned}$$

$$\psi(0, u) = u$$

In Equation (8), $k_u(\psi_x)$ represents $\mathbb{E}[e^{\psi_x J_u} - 1]$ and $k_d(-\psi_x)$ represents $\mathbb{E}[e^{-\psi_x J_d} - 1]$. By assuming the domain of positive and negative exponential jump with γ_u and γ_d rate parameters, for $k_u(\psi_x)$ and $k_d(-\psi_x)$, we have:

$$\begin{aligned} k_u(\psi_x) &= \int_0^\infty (e^{\psi_x J_u} - 1) f(J_u) dJ_u \\ &= \int_0^\infty (e^{\psi_x J_u} - 1) \gamma_u e^{-\gamma_u J_u} dJ_u = \frac{\psi_x}{\gamma_u - \psi_x} \quad (10) \end{aligned}$$

$$\begin{aligned} k_d(-\psi_x) &= \int_0^\infty (e^{-\psi_x J_d} - 1) f(J_d) dJ_d \\ &= \int_0^\infty (e^{-\psi_x J_d} - 1) \gamma_d e^{-\gamma_d J_d} dJ_d = \frac{-\psi_x}{\gamma_d + \psi_x} \end{aligned}$$

Considering $k_u(\psi) = \frac{\psi}{\gamma_u - \psi}$, $k_d(-\psi) = \frac{-\psi}{\gamma_d + \psi}$ and $\psi(T-t, u) = u e^{-\kappa(T-t)}$, the answer for differential Equation (8) is as follows:

$$\begin{aligned} \phi(T-t, u) &= \mu^* (1 - e^{-\kappa(T-t)}) \\ &\quad + \frac{\sigma^2 u^2}{4\kappa} (1 - e^{-2\kappa(T-t)}) \\ &\quad + \frac{\eta_u}{\kappa} \ln \frac{\gamma_u - u e^{-\kappa(T-t)}}{\gamma_u - u} \quad (11) \\ &\quad + \frac{\eta_d}{\kappa} \ln \frac{\gamma_d + u e^{-\kappa(T-t)}}{\gamma_d + u} \end{aligned}$$

Finally, with $\phi(T-t, u)$ and $\psi(T-t, u)$, the one factor futures price with exponential jump by placing $z = -i$ or $u = iz = 1$ will be defined as:

$$\begin{aligned} F(T, S_t) &= \mathbb{E}^Q [S_T | \mathcal{F}_t] = \mathbb{E}^Q [e^{X_T} | \mathcal{F}_t] \\ &= \exp(\phi(T-t, 1) + X_t \psi(T-t, 1)) \quad (12) \\ &= \exp[e^{-\kappa(T-t)} \ln S_t + \mu^* (1 - e^{-\kappa(T-t)})] \end{aligned}$$

$$+ \frac{\sigma^2}{4k} (1 - e^{-2\kappa(T-t)}) + \frac{\eta_u \ln \frac{\gamma_u - e^{-\kappa(T-t)}}{\gamma_u - 1}}{\kappa} + \frac{\eta_d \ln \frac{\gamma_d + e^{-\kappa(T-t)}}{\gamma_d + 1}}{\kappa}$$

Equation (12) determines the futures contract price at time t for maturity of T . This relation is a closed form answer for stochastic differential equation that brought up in one- factor model with exponential stochastic jump. In this relation, futures price is a function of current spot price and time remaining to maturity. This price is a risk neutral valuation while was extracted in the absence of arbitrage condition².

3.2. Two-factor model of futures pricing with spot price jump (exponential jump domain)

Two factor model of futures pricing in diffusion-jump framework is an extension of Gibson-Schwartz (1991) and Schwartz(1997) two- factor model. In this model, spot price dynamics and convenience yield are used to determine futures price. According to the convenience yield, keeping physical commodity has some profits for trader while futures contract does not have this feature. As an indicator, the net flow of storage commodity benefits in per unit of time can be called convenience yield. This flow of gains has an effect on pricing of commodity futures contract (Hull, 2012).

If (Ω, \mathcal{F}, P) is probability space and it is assumed that spot price of commodity follows Geometric Brownian jump-diffusion process and The convenience yield follows mean reversion process, then, we define the followings:

$$dS_t = (\mu - C_t)S_t dt + \sigma_1 S_t dB_{1t} + S_t(e^{J_u} - 1)dN_{u,t} + S_t(e^{-J_d} - 1)dN_{d,t} \quad (13)$$

and

$$dC_t = \theta(\alpha - C_t)dt + \sigma_2 dB_{2t} \quad (14)$$

Equation (13) shows stochastic behavior of spot price in terms of jump-diffusion process while diffusion part is expresses by Geometric Brownian process that adjusted respect to convenience yield. Parameter μ is positive constant that shows expected return of commodity prices per unit of time. μ is adjusted by stochastic convenience yield. In futures contract, convenience yield is a negative factor in pricing and they reduce value of commodity futures contract. The σ_1 indicates standard deviation of return on commodity spot price per

unit of time, also this parameter is constant as well, and B_{1t} is standard Brownian motion.

$S_t(e^{J_u} - 1)dN_{u,t}$ represents the process of increasing jump that follows the Poisson process $N_{u,t}$ with rate η_u . the domain of increasing stochastic spot price jumps is shown by J_u that follows the exponential probability distribution with rate parameter γ_u . $S_t(e^{-J_d} - 1)dN_{d,t}$ is a decreasing jump process with a Poisson process $N_{d,t}$ with rate η_d . Also, the absolute value of the domain of spot price stochastic reduction jump is shown by J_d that follows the exponential probability distribution with parameter γ_d .

Equation (14) explains stochastic behavior of convenience yield by using mean reversion process. In this process, C_t shows convenience yield at time t , α indicates the degree of mean reversion, $\theta > 0$ is the speed of adjustment, σ_2 is the convenience yield standard deviation from the mean value and B_{2t} is standard Brownian process in stochastic behavior of convenience yield. Also, increments of Brownian motions B_{1t} and B_{2t} are correlated with coefficient ρ that is shown as $dB_{1t} \cdot dB_{2t} = \rho dt$.

Equivalent Martingale measure is used to convert model to a no-arbitrage model. To this purpose $dB_{1t} = dB_{1t}^* - \frac{\mu-r}{\sigma_1} dt$ is used to diffusion-jump process of Equation (15). In two-factor model, commodity is similar to an asset which its owner, receives stochastic profit C_t . So, drift term of commodity price process is adjusted respect to risk equals to $r - C_t$. Actually, if there is no futures contract or commodity purchasing, trader can invest safety and receive no- risk interest rate r , but loses convenience yield (Bjersund, 1991). Thus, it expects that mean value of spot price process is $r - C_t$.

Finally, by replacing above term in stochastic differential Equation (13), new spot price equation in the probability space Q can be defined as:

$$dS_t = (r - C_t)S_t dt + \sigma_1 S_t dB_{1t}^* + S_t(e^{J_u} - 1)dN_{u,t} + S_t(e^{-J_d} - 1)dN_{d,t} \quad (15)$$

Additionally, using $B_{2t} = dB_{2t}^* - \lambda dt$, to make no- arbitrage condition in the diffusion process of convenience yield, implies:

$$dC_t = \theta(\alpha^* - C_t) dt + \sigma_2 dB_{2t}^* \quad (16)$$

$$\alpha^* = \alpha - \frac{\sigma_2 \lambda}{\theta}, \quad dB_{1t}^* \cdot dB_{2t}^* = \rho dt$$

Assuming $X_t = \ln S_t$ and using the jump-diffusion Ito's lemma for (17), we can write:

$$dX_t = \left(r - \frac{\sigma_1^2}{2} - C_t \right) dt + \sigma_1 dB_{1t}^* + J_u dN_{u,t} - J_d dN_{d,t} \quad (17)$$

² The above futures price extracted by referring to Ross(1995), Duffie et al. (2000), Dempster et al. (2010), Villaplana (2003), Carlea and Figueroa (2005), Bjersund (1991), Schwartz (1997), Bjork (2003), Tankov and Rama (2004).

Like one-factor model, two-factor Affine jump-diffusion model has exponential Affine characteristic function at maturity time T respect to initial value X_t and C_t :

$$\mathbb{E}^Q [e^{u_1 X_T + u_2 C_T} | \mathcal{F}_t] = \exp[\phi(T-t, u) + X_t \psi_X(T-t, u) + C_t \psi_C(T-t, u)] \quad (18)$$

Where $u = (u_1, u_2)$ and Riccati equation equals to:

$$\dot{\psi}_X(T-t, u) = 0, \quad \psi_X(0, u) = u_1 \quad (19)$$

$$\dot{\psi}_C(T-t, u) = -\psi_C - \theta \psi_C, \quad \psi_C(0, u) = u_2 \quad (20)$$

$$\begin{aligned} \dot{\phi}(T-t, u) &= r\psi_X - \frac{\sigma_1^2}{2}\psi_X + \theta\alpha^* \psi_C \\ &+ \frac{1}{2}\psi_X^2 \sigma_1^2 + \sigma_1 \sigma_2 \rho \psi_X \psi_C + \frac{1}{2}\sigma_2^2 \psi_C^2 \\ &+ \eta_u k_u(\psi_X) + \eta_d k_d(-\psi_X), \phi(0, u) \\ &= 0 \end{aligned} \quad (21)$$

Like one-factor model $k(\psi_X)$ and $k_d(-\psi_X)$ are equal to:

$$k_u(\psi_X) = \frac{\psi_X}{\gamma_u - \psi_X}, \quad k_d(-\psi_X) = \frac{-\psi_X}{\gamma_d + \psi_X} \quad (22)$$

By replacing Equation (22) in Equation (21), the solutions of Riccati differential equations are obtained respectively:

$$\psi_X(T-t, u) = u_1 \quad (23)$$

$$\psi_C(T-t, u) = \frac{-u_1}{\theta} + \left(\frac{u_1}{\theta} + u_2\right) e^{-\theta t} \quad (24)$$

and

$$\begin{aligned} \phi(T-t, u) &= \left[\left(r - \frac{\sigma_1^2}{2} - \alpha^* \right) u_1 \right. \\ &\quad \left. + \frac{u_1^2 \sigma_1^2}{2} + \frac{u_1^2 \sigma_2^2}{2\theta^2} \right. \\ &\quad \left. - \frac{u_1^2 \sigma_1 \sigma_2 \rho}{\theta} + \frac{\eta_u u_1}{\gamma_u - u_1} - \frac{\eta_d u_1}{\gamma_d + u_1} \right] (T-t) \\ &+ \left(\alpha^* + \frac{u_1 \sigma_1 \sigma_2 \rho}{\theta} - \frac{u_1 \sigma_2^2}{\theta^2} \right) \left(\frac{u_1}{\theta} + u_2 \right) (1 \\ &\quad - e^{-\theta(T-t)}) \\ &+ \frac{\sigma_2^2}{4\theta} \left(\frac{u_1}{\theta} + u_2 \right)^2 (1 - e^{-2\theta(T-t)}) \end{aligned} \quad (25)$$

By replacing Equations (23), (24) and (25) in Equation (18) and also placing $u_1 = 1$ and $u_2 = 0$, the two-factor futures price by considering exponential jump is defined as:

$$F(T, S_t, C_t) = \mathbb{E}^Q [S_T | \mathcal{F}_t] = \mathbb{E}^Q [e^{X_T} | \mathcal{F}_t] \quad (26)^3$$

$$\begin{aligned} &= S_t \exp \left[\left(r - \alpha + \frac{\lambda \sigma_2}{\theta} + \frac{\sigma_2^2}{2\theta^2} - \frac{\sigma_1 \sigma_2 \rho}{\theta} \right. \right. \\ &\quad \left. \left. + \frac{\eta_u}{\gamma_u - 1} \right. \right. \\ &\quad \left. \left. - \frac{\eta_d}{\gamma_d + 1} \right) (T-t) + \frac{\sigma_2^2}{4\theta^3} (1 - e^{-2\theta(T-t)}) \right. \\ &\quad \left. + \left(\frac{\sigma_1 \sigma_2 \rho}{\theta^2} - \frac{\sigma_2^2}{\theta^3} + \frac{\alpha}{\theta} - \frac{C_t}{\theta} - \frac{\lambda \sigma_2}{\theta^2} \right) (1 \right. \\ &\quad \left. - e^{-\theta(T-t)}) \right] \end{aligned}$$

In compared with futures price relationship of Schwartz (1997) without jump two-factor model, commodity futures price in Equation (26) and Schwartz Futures Price are completely the same except for jump term which is added to Equation (26).

3.3. Three-Factor Model of Futures Pricing with Spot Price Jump

The three-factor model in the framework of jump-diffusion is the development of Schwartz (1997) three-factor model. In this model, stochastic spot price follows geometric Brownian jump-diffusion process. In addition, convenience yield and instantaneous interest rate are considered as stochastic factor which follows mean reversion process:

$$dS_t = (\mu - C_t) S_t dt + \sigma_1 S_t dB_{1t} + S_t (e^{J_u} - 1) dN_{u,t} + S_t (e^{-J_d} - 1) dN_{d,t} \quad (27)$$

$$dC_t = \theta(\alpha - C_t) dt + \sigma_2 dB_{2t} \quad (28)$$

$$d\tau_t = \omega(\pi - \tau_t) dt + \sigma_3 dB_{3t} \quad (29)$$

$$\begin{aligned} dB_{1t} \cdot dB_{2t} &= \rho_1 dt, \quad dB_{2t} \cdot dB_{3t} \\ &= \rho_2 dt, \quad dB_{1t} \cdot dB_{3t} = \rho_3 dt \end{aligned} \quad (30)$$

Equation (27) shows behavior of spot price in form of jump-diffusion process which diffusion term is explained by Geometric Brownian that adjusted by commodity convenience yield. The terms of $S_t (e^{J_u} - 1) dN_{u,t}$ and $S_t (e^{-J_d} - 1) dN_{d,t}$ show increase and decrease jump like two previous models. Similar to two-factor model, the stochastic behavior of convenience yield is explained by mean reversion process. In this process, C_t is convenience yield at time t , α shows degree of mean reversion, $\theta > 0$, σ_2 and B_{2t} respectively are the speed of adjustment, standard deviation of convenience yield and standard Brownian motion in behavior of convenience yield.

Also, stochastic behavior of interest rate is explained by mean reversion process which in this process, τ_t is instantaneous interest rate at time t , π shows degree of mean reversion

³ The above two-factor future price has been extracted from referring to Duffie et al. (2000), Dempster et al. (2010), Villaplana (2003), Carlea and Figueroa (2005), Schwartz (1997), Bjork (2003), Tankov (2004).

parameter, $\omega > 0$, σ_3 and B_{3t} respectively are the speed of adjustment, standard deviation of instantaneous interest rate from mean and standard Brownian motion in behavior of instantaneous interest rate.

To convert the model to an no-arbitrage model, equivalent Martingale measure is required. In order to achieve this purpose, $dB_{1t} = dB_{1t}^* - \frac{\mu - r_t}{\sigma_1} dt$ is used to show jump-diffusion process of spot price.

$$dS_t = (r - C_t)S_t dt + \sigma_1 S_t dB_{1t}^* + S_t(e^{J_u} - 1)dN_{u,t} + S_t(e^{-J_d} - 1)dN_{d,t} \quad (31)$$

If λ_2 is the market value of a unit of convenience yield risk, $dB_{2t} = dB_{2t}^* - \lambda_2 dt$ provide absence of arbitrage condition in the process of diffusion convenience yield is used:

$$dC_t = \theta \left(\alpha - \frac{\sigma_2 \lambda_2}{\theta} - C_t \right) dt + \sigma_2 dB_{2t}^* \quad (32)$$

Finally, λ_3 is the market value of a unit of interest rate risk to convert interest rate process based on equivalent Martingale measure. By replacing $dB_{3t} = dB_{3t}^* - \lambda_3 dt$ in stochastic differential equilibrium of interest rate, implies:

$$dr_t = \omega \left(\pi - \frac{\sigma_3 \lambda_3}{\omega} - r_t \right) dt + \sigma_3 dB_{3t}^* \quad (33)$$

In addition, correlation coefficients of Brownian increment based on Q probability measure are equal to:

$$dB_{1t}^* \cdot dB_{2t}^* = \rho_1 dt, \quad dB_{2t}^* \cdot dB_{3t}^* = \rho_2 dt, \quad dB_{1t}^* \cdot dB_{3t}^* = \rho_3 dt \quad (34)$$

If $X_t = \ln S_t$, by using Ito's jump-diffusion lemma to Equation (31), we can write:

$$dX_t = \left(r - \frac{\sigma_1^2}{2} - C_t \right) dt + \sigma_1 dB_{1t}^* + J_u \epsilon - J_d dN_{d,t} \quad (35)$$

Similar to one-factor and two-factor models, three-factor Affine jump-diffusion model has exponential Affine characteristic function at maturity time T respect to initial value X_t and C_t :

$$\mathbb{E}^Q [e^{u_1 X_T + u_2 C_T + u_3 r_T} | \mathcal{F}_t] = \exp[\phi(T-t, u) + X_t \psi_X(T-t, u) + C_t \psi_C(T-t, u) + r_t \psi_r(T-t, u)] \quad (36)$$

where $u = (u_1, u_2, u_3)$

The function of $g(t, X_t, C_t, r_t)$ is defined in the following framework in order to imply better understanding the process of pricing and finding

Riccati differential equations:

$$g(t, X_t, C_t, r_t) = \exp[\phi(T-t, u) + X_t \psi_X(T-t, u) + C_t \psi_C(T-t, u) + r_t \psi_r(T-t, u)] \quad (37)$$

By applying Ito's lemma to $g(t, X_t, C_t, r_t)$, the process implies:

$$g(T, X_T, C_T, r_T) - g(t, X_t, C_t, r_t) = -(\dot{\phi} + \dot{\psi}_X X_t + \dot{\psi}_C C_t + \dot{\psi}_r r_t) g dt + \left(r_t - \frac{\sigma_1^2}{2} - C_t \right) \psi_X g dt + \theta \left(\alpha - \frac{\sigma_2 \lambda_2}{\theta} - C_t \right) \psi_C g dt + \omega \left(\pi - \frac{\sigma_3 \lambda_3}{\omega} - r_t \right) \psi_r g dt + \frac{1}{2} \sigma_1^2 \psi_X^2 g dt + \frac{1}{2} \sigma_2^2 \psi_C^2 g dt + \frac{1}{2} \sigma_3^2 \psi_r^2 g dt + \sigma_1 \sigma_2 \rho_1 \psi_X \psi_C g dt + \sigma_2 \sigma_3 \rho_2 \psi_C \psi_r g dt + \sigma_1 \sigma_3 \rho_3 \psi_X \psi_r g dt + \sigma_1 \psi_X g dB_{1t}^* + \sigma_2 \psi_C g dB_{2t}^* + \sigma_3 \psi_r g dB_{3t}^* + [g(t, X_t + J_u, C_t, r_t) - g(t, X_t, C_t, r_t)] + [g(t, X_t - J_d, C_t, r_t) - g(t, X_t, C_t, r_t)] \quad (38)$$

In order to find Riccati differential equations, it is necessary that $g(t, X_t, C_t, r_t)$ to be as a Martingale or equivalently $\mathbb{E}^Q [g(T, X_T, C_T, r_T) | \mathcal{F}_t] = g(t, X_t, C_t, r_t)$.

Therefore, we calculate expected value of (38) respect to the equivalent probability measure. For this purpose, it is necessary that the expected value of right side of Equation (38) be zero. Thus, in this way, we can define the following Riccati differential equations as:

$$\dot{\psi}_X(T-t, u) = 0, \quad \psi_X(0, u) = u_1 \quad (39)$$

$$\dot{\psi}_C(T-t, u) = -\psi_X - \theta \psi_C, \quad \psi_C(0, u) = u_2 \quad (40)$$

$$\dot{\psi}_r(T-t, u) = \psi_X - \omega \psi_r, \quad \psi_r(0, u) = u_3 \quad (41)$$

$$\begin{aligned} \dot{\phi}(T-t, u) = & -\frac{\sigma_1^2}{2} \psi_X^2 + \theta \left(\alpha - \frac{\sigma_2 \lambda_2}{\theta} \right) \psi_C + \omega \left(\pi - \frac{\sigma_3 \lambda_3}{\omega} \right) \psi_r + \frac{1}{2} \sigma_1^2 \psi_X^2 + \frac{1}{2} \sigma_2^2 \psi_C^2 + \frac{1}{2} \sigma_3^2 \psi_r^2 + \sigma_1 \sigma_2 \rho_1 \psi_X \psi_C + \sigma_2 \sigma_3 \rho_2 \psi_C \psi_r + \sigma_1 \sigma_3 \rho_3 \psi_X \psi_r + \eta_u k_u(\psi_X) + \eta_d k_d(-\psi_X) \end{aligned} \quad (42)$$

$$\phi(0, u) = 0$$

By solving Equations (39), (40) and (41) we can get:

$$\psi_x(T-t, u) = u_1 \quad (43)$$

$$\psi_c(T-t, u) = \frac{-u_1}{\theta} + \left(\frac{u_1}{\theta} + u_2\right) e^{-\theta t} \quad (44)$$

$$\psi_r(T-t, u) = \frac{u_1}{\omega} + \left(\frac{-u_1}{\omega} + u_3\right) e^{-\omega t} \quad (45)$$

Similar to one-factor and two factor models, $k(\psi_x)$ and $k_d(-\psi_x)$ are equal to:

$$k_u(\psi_x) = \frac{\psi_x}{\gamma_u - \psi_x}, \quad k_d(-\psi_x) = \frac{-\psi_x}{\gamma_d + \psi_x} \quad (46)$$

Equation (42) is solved by using Equations (43), (44), (45) and (46). But, we disregard to show the algebraic operations. Finally, the three-factor pricing equation is extracted like one and two-factor model. For this purpose, Equations (43), (44), (45) and $\phi(T-t, u)$ are replaced in Equation (36) and considering $u_1 = 1$, $u_2 = 0$ and $u_3 = 0$; thus we have:

$$\begin{aligned} F(T, S_t, C_t, r_t) &= \mathbb{E}^Q[S_T | \mathcal{F}_t] = \mathbb{E}^Q[e^{x_T} | \mathcal{F}_t] \\ &= S_t \exp \left[\left(\pi - \frac{\lambda_3 \sigma_3}{\omega} - \alpha + \frac{\lambda_2 \sigma_2}{\theta} + \frac{\sigma_2^2}{2\theta^2} \right. \right. \\ &\quad \left. \left. + \frac{\sigma_3^2}{2\omega^2} \right. \right. \\ &\quad \left. \left. - \frac{\sigma_1 \sigma_2 \rho_1}{\theta} + \frac{\sigma_1 \sigma_3 \rho_3}{\omega} - \frac{\sigma_2 \sigma_3 \rho_2}{\theta \omega} + \frac{\eta_u}{\gamma_u - 1} - \frac{\eta_d}{\gamma_d + 1} \right. \right. \\ &\quad \left. \left. - t \right) \right. \\ &\quad \left. + \left(\frac{\alpha - C_t}{\theta} - \frac{\lambda_2 \sigma_2}{\theta^2} + \frac{\sigma_1 \sigma_2 \rho_1}{\theta^2} - \frac{\sigma_2^2}{\theta^3} + \frac{\sigma_2 \sigma_3 \rho_2}{\omega \theta^2} \right) (1 - e^{-t}) \right. \\ &\quad \left. + \left(-\frac{\pi}{\omega} + \frac{r_t}{\omega} + \frac{\lambda_3 \sigma_3}{\omega^2} - \frac{\sigma_1 \sigma_3 \rho_3}{\omega^2} - \frac{\sigma_3^2}{\omega^3} + \frac{\sigma_2 \sigma_3 \rho_2}{\theta \omega^2} \right) (1 - e^{-\omega(T-t)}) \right. \\ &\quad \left. + \frac{\sigma_2^2}{4\theta^3} (1 - e^{-2\theta(T-t)}) + \frac{\sigma_3^2}{4\omega^3} (1 - e^{-2\omega(T-t)}) \right. \\ &\quad \left. - \frac{2\sigma_2 \sigma_3 \rho_2}{\theta \omega (\theta + \omega)} (1 - e^{-(\theta + \omega)(T-t)}) \right] \end{aligned} \quad (47)$$

Commodity futures price show in Equation (47) is the same as three-factor model pricing of Schwartz (1997), but jump term make some changes.

The above future price is function of spot price, convenience yield, instantaneous interest rate and time to maturity⁴.

4. Conclusion

Derivatives are referred to a set of tradable tools in financial markets which achieve their features from underlying asset. Derivatives instruments are an alternative way in simple buying and selling to control for the price volatility risks. Accordingly, conducting a study on pricing of

futures can provide useful tools to understand the stochastic behavior of prices to management the price volatility risk. Thus, this study has reviewed valuation of commodity futures contracts by considering Ross (1995) one-factor futures pricing model as a function of spot price, Gibson and Schwartz (1990) two-factor futures pricing model as a function of spot price and convenience yield and finally Schwartz (1997) three-factor futures pricing model as a function of spot price, convenience yield and instantaneous interest rate by adding jump to stochastic behavior of commodity spot price. For this purpose, in all recommended basic models, it was assumed the spot price which follows Jump-diffusion stochastic process, with an exponential probability distribution of jump domain.

Finally, commodity futures price framework in three basic models have been presented as a function of spot price and time to maturity in one-factor model, spot price, convenience yield and time to maturity in two-factor model, spot price, convenience yield, instantaneous interest rate and time to maturity in three-factor model by using affine jump-diffusion approach.

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